Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Linear Algebra and its Applications 420 (2007) 663–666

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

L_p Linear discrepancy of totally unimodular matrices

Benjamin Doerr *

Max-Planck-Institut für Informatik, Stuhlsatzenhausweg 85, 66123 Saarbrücken, Germany

Received 24 June 2003; accepted 28 August 2006
Available online 9 October 2006
Submitted by R.A. Brualdi

Abstract

Let $p \in [1, \infty[$ and $c_p = \max_{a \in [0,1]}((1-a)a^p + a(1-a)^p)^{1/p}$. We prove that the known upper bound $\operatorname{lindisc}_p(A) \leqslant c_p$ for the L_p linear discrepancy of a totally unimodular matrix A is asymptotically sharp, i.e.,

$$\sup_A \operatorname{lindisc}_p(A) = c_p.$$

We estimate $c_p = \frac{p}{p+1}\left(\frac{1}{p+1}\right)^{1/p}(1+\varepsilon_p)$ for some $\varepsilon_p \in \left[0, 2^{-p+2}\right]$, hence $c_p = 1 - \frac{\ln p}{p}(1+o(1))$. We also show that an improvement for smaller matrices as in the case of L_∞ linear discrepancy cannot be expected. For any $p \in \mathbb{N}$ we give a totally unimodular $(p+1) \times p$ matrix having L_p linear discrepancy greater than $\frac{p}{p+1}\left(\frac{1}{p+1}\right)^{1/p}$.

AMS classification: 11K38

Keywords: Linear discrepancy; Totally unimodular matrix; Rounding; Halftoning

1. Introduction and results

Let $p \in [1, \infty[$ and let $A \in \mathbb{R}^{m \times n}$. Denote the rows of A by $a^{(1)}, \ldots, a^{(m)} \in \mathbb{R}^n$. Let $x \in [0, 1]^n$. The L_p *linear discrepancy of A with respect to x* is

* Tel.: +49 6819325104; fax: +49 6819325199.

0024-3795/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2006.08.019

$$\operatorname{lindisc}_p(A, x) = \min_{y \in \{0,1\}^n} \frac{1}{m^{1/p}} \|A(x-y)\|_p = \min_{y \in \{0,1\}^n} \left(\frac{1}{m} \sum_{i=1}^{m} |a^{(i)} \cdot (x-y)|^p \right)^{1/p}.$$

The L_p linear discrepancy of A is $\operatorname{lindisc}_p(A) = \max_{x \in [0,1]^n} \operatorname{lindisc}_p(A, x)$.

The matrix A is called *totally unimodular*, if each square submatrix has determinant -1, 0 or 1. In particular, the entries of a totally unimodular matrix are from $\{-1, 0, 1\}$. Put $c_p = \max_{a \in [0,1]}((1-a)a^p + a(1-a)^p)^{1/p}$. Motivated by an application in image processing, Asano et al. [2] (cf. also the survey Asano [1]) show and estimate

$$\operatorname{lindisc}_p(A) \leqslant c_p \leqslant 1 - \frac{1}{p+1}. \tag{1}$$

They also note that the n-dimensional identity matrix I_n satisfies $\operatorname{lindisc}_p(I_n) = \frac{1}{2}$ for all p and n. This shows that the first inequality in (1) is sharp for $p \leqslant 3$.

The objective of this note is to improve the lower bound for $p \geqslant 3$. We show that for all $n \in \mathbb{N}$, there is a totally unimodular matrix $A \in \{0, 1\}^{(n+1)\times n}$ such that

$$\operatorname{lindisc}_p(A) \geqslant c_p(1 + \mathrm{o}(1))$$

with o(1) term depending on n only. Estimating c_p tighter than in [2] yields

$$\frac{p}{p+1}\left(\frac{1}{p+1}\right)^{1/p} \leqslant c_p \leqslant \frac{p}{p+1}\left(\frac{1}{p+1}\right)^{1/p}(1 + 2^{-p+2}).$$

Thus $c_p = 1 - \frac{\ln p}{p}(1 + \mathrm{o}(1))$. Finally, we give for any $p \in \mathbb{N}$ a totally unimodular $(p+1) \times p$ matrix A such that $\operatorname{lindisc}_p(A) > \frac{p}{p+1}\left(\frac{1}{p+1}\right)^{1/p}$. This shows that an improvement of (1) for smaller matrices as was recently proven for the L_∞ linear discrepancy (see Section 3) cannot exist.

2. Proofs of the main results

Note first that $c_p = \max_{a \in [0, \frac{1}{2}]}((1-a)a^p + a(1-a)^p)^{1/p}$ due to symmetry. Since the case $p \leqslant 3$ was already completely solved in [2], we assume $p \geqslant 3$ in the following. We prove the following lower bound.

Theorem 1. *For all $p \geqslant 3$ and all $n \in \mathbb{N}$, there is a totally unimodular matrix $A \in \{0, 1\}^{(n+1)\times n}$ such that*

$$\operatorname{lindisc}_p(A) \geqslant c_p(1 + \mathrm{o}(1))$$

with o(1) *term depending on n only.*

Proof. Let $a \in \left]0, \frac{1}{2}\right]$ and $n \in \mathbb{N}$ be sufficiently large. Define $A \in \{0, 1\}^{(n+1)\times n}$ by $a_{ij} = 1$ if and only if $i = j$ or $i = n + 1$. Clearly, A is totally unimodular. Let $x = a\mathbf{1}_n \in \mathbb{R}^n$. Let $y \in \{0, 1\}^n$ such that $\|A(x - y)\|_p$ is minimal. Let k be the number of $i \in \{1, \dots, n\}$ such that $y_i = 1$. Then

$$\|A(x-y)\|_p^p = (n-k)a^p + k(1-a)^p + |na - k|^p =: f(k).$$

Note that this value only depends on the number k, but not on the distribution of the ones in y. f viewed as function on the reals is convex and has a minimum at $k_0 = an - \left(\frac{(1-a)^p - a^p}{p}\right)^{1/(p-1)}$. This yields

$$\begin{aligned}\|A(x-y)\|_p^p &\geqslant n((1-a)a^p + a(1-a)^p) - ((1-a)^p - a^p)^{p/(p-1)} \frac{p-1}{p^{p/(p-1)}}\\ &\geqslant n((1-a)a^p + a(1-a)^p) - 1.\end{aligned}$$

Thus

$$\operatorname{lindisc}_p(A) \geqslant \max_{a\in[0,1]} \sqrt[p]{\frac{n}{n+1}((1-a)a^p + a(1-a)^p) - \frac{1}{n+1}} = c_p(1+\mathrm{o}(1)). \qquad \square \tag{2}$$

Having shown that c_p is the supremum L_p linear discrepancy of a totally unimodular matrix, we now estimate the constant c_p.

Lemma 2. *For* $p \geqslant 3$,

$$\frac{p}{p+1}\left(\frac{1}{p+1}\right)^{1/p} \leqslant c_p \leqslant \frac{p}{p+1}\left(\frac{1}{p+1}\right)^{1/p}(1+2^{-p+2}).$$

In particular, $c_p = 1 - \frac{\ln p}{p}(1+\mathrm{o}(1))$.

Proof. We use the estimate $1+x \leqslant \mathrm{e}^x \leqslant 1+\frac{x}{1-x}$ valid for all $x<1$ several times. Putting $a = \frac{1}{p+1}$ in the definition of c_p, we obtain

$$c_p \geqslant \left(\frac{p^p+p}{(p+1)^{p+1}}\right)^{1/p} > \frac{p}{p+1}\left(\frac{1}{p+1}\right)^{1/p}.$$

For the upper bound, we have

$$\begin{aligned}c_p &= \left(\max_{a\in[0,\frac{1}{2}]}((1-a)a^p + a(1-a)^p)\right)^{1/p} \leqslant \left(\max_{a\in[0,\frac{1}{2}]}(1-a)a^p + \max_{a\in[0,\frac{1}{2}]} a(1-a)^p\right)^{1/p}\\ &= \left(\frac{p^p}{(p+1)^{p+1}} + 2^{-p-1}\right)^{1/p}.\end{aligned}$$

Since $x \mapsto x^{1/p}$ is concave for $p \geqslant 1$, we conclude

$$\begin{aligned}c_p &\leqslant \left(\frac{p^p}{(p+1)^{p+1}}\right)^{1/p} + \frac{1}{p}\left(\frac{p^p}{(p+1)^{p+1}}\right)^{-1+(1/p)} 2^{-p}\\ &= \frac{p}{p+1}\left(\frac{1}{p+1}\right)^{1/p}\left(1+\left(\frac{p+1}{p}\right)^{p+1}2^{-p}\right).\end{aligned}$$

Finally, we estimate

$$\left(\frac{p+1}{p}\right)^{p+1} 2^{-p} \leqslant \mathrm{e}^{(p+1)/p}2^{-p} \leqslant \mathrm{e}^{4/3}2^{-p} \leqslant 4\cdot 2^{-p}$$

using our assumption $p \geqslant 3$. From $\frac{p}{p+1}\left(\frac{1}{p+1}\right)^{1/p} = 1 - \frac{\ln p}{p}(1+\mathrm{o}(1))$ we derive the second claim. $\square$

3. Improving the $1 + o(1)$ term relative to n?

We now turn to the dependency of n again. Recent results concerning the L_∞ linear discrepancy make this a natural problem. The *L_∞ linear discrepancy of A with respect to x* is

$$\operatorname{lindisc}_\infty(A, x) = \min_{y \in \{0,1\}^n} \|A(x - y)\|_\infty,$$

the *L_∞ linear discrepancy of A* is $\operatorname{lindisc}_\infty(A) = \max_{x\in[0,1]^n} \operatorname{lindisc}_\infty(A, x)$.

For the L_∞ linear discrepancy of totally unimodular matrices, the bound $\operatorname{lindisc}_\infty(A) \leqslant 1$ is well known and follows easily from the theorem of Hoffman and Kruskal [6] (similar to the proof of the upper bound for the L_p linear discrepancy in [2]). However, this bound is not sharp. If an $m \times n$ matrix A is totally unimodular, then $\operatorname{lindisc}_\infty(A) \leqslant \frac{n}{n+1}$ was shown in [4,5] and independently in [3]. A similar improvement, as could be conjectured from equation (2), is not possible for the L_p linear discrepancy:

Theorem 3. *For any $p \in \mathbb{N}$, there is a totally unimodular matrix A having p columns only such that* $\operatorname{lindisc}_p(A) > \frac{p}{p+1}\left(\frac{1}{p+1}\right)^{1/p}$.

Note that the bound above is less than a factor of $1 + 2^{-p+2}$ below our upper bound. We stated this result for integral p only to keep things simple. However, it is not difficult to see that similar statements can be made for arbitrary p.

Proof of Theorem 3. Let $p \in \mathbb{N}$. Let $\lambda \in \mathbb{N}_0$ and $n = \lambda(p + 1) + p$. Let A be the $(n + 1) \times n$ matrix defined in the proof of Theorem 1. Let $a = \frac{1}{p+1}$ and $y \in \{0, 1\}^n$ such that $\|A(a\mathbf{1}_n - y)\|_p$ is minimal. Now $f(k)$ defined as in the proof of Theorem 1 but viewed as mapping from the integers is minimal for $k = \lfloor na \rfloor = na - \frac{p}{p+1}$ and $k = \lceil na \rceil = na + \frac{1}{p+1}$. In both cases we have $f(k) = (n + 1)\frac{p^p+p}{(p+1)^{p+1}}$, i.e.,

$$\operatorname{lindisc}_p(A, a\mathbf{1}_n) = \left(\frac{p^p + p}{(p + 1)^{p+1}}\right)^{1/p} > \frac{p}{p + 1}\left(\frac{1}{p + 1}\right)^{1/p}. \qquad \square$$

References

[1] T. Asano, Digital halftoning: algorithm engineering challenges, IEICE Trans. Inf. Syst. E86-D (2003) 159–178.

[2] T. Asano, N. Katoh, K. Obokata, T. Tokuyama, Matrix rounding under the L_p-discrepancy measure and its application to digital halftoning, SIAM J. Comput. 32 (2003) 1423–1435.

[3] T. Bohman, R. Holzman, Linear versus hereditary discrepancy, Combinatorica 25 (2005) 39–47.

[4] B. Doerr, Lattice approximation and linear discrepancy of totally unimodular matrices, in: Proceedings of the 12th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), 2001, pp. 119–125.

[5] B. Doerr, Linear discrepancy of totally unimodular matrices, Combinatorica 24 (2004) 117–125.

[6] A.J. Hoffman, J.B. Kruskal, Integral boundary points of convex polyhedra, in: H.W. Kuhn, A.W. Tucker (Eds.), Linear Inequalities and Related Systems, 1956, pp. 223–246.